Lecture: 3-4 The Chain Rule

When you have a function that is a composite function, like $y = \sqrt{x^2+1}$, the formulas we have so far do not let us find y'. However, if you write your composite function as $f \circ g$, we have a formula for the derivative.

The Chain Rule: If f and g are differentiable and $F = f \circ g$, then F is differentiable and

$$F'(x) = f'(g(x))g'(x).$$

Example 1: Write the composite function in the form $f(g(x))$ and then find y'.

(a) $y = (1+3x)^9$

(b) $y = \dfrac{1}{(x^2+2x-5)^9}$

Example 2: Write the composite function in the form $f(g(x))$. Then, find y'.

(a) $y = \cos(x^3)$

(b) $y = \cos^3(x)$

Example 3: Find the derivative of $f(x) = \left(\frac{x+5}{2x-1}\right)^5$.

Example 4: Find the derivative of $f(x) = (2x-1)^6(x^3-2x+1)^3$

Example 5: Find the derivative of the following functions.

(a) $y = e^{x \sec x}$

(b) $y = \sin(\sin(\sin x))$

Review: The Chain Rule: If f and g are differentiable and $F = f \circ g$, then F is differentiable and

$$F'(x) = \underline{\hspace{3cm}}$$

Example 6: Let $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g'(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

Example 7: Find the derivatie of the following functions.

(a) $g(x) = \sqrt[5]{x^3 - 1}$

(b) $h(x) = \sin^5(4x^2)$

Formula: Derivative of $y = b^x$ $\dfrac{d}{dx}(b^x) = (\ln b)b^x$

Example 8: Find the derivative of the following functions.

(a) $y = 5^x$

(b) $f(x) = 10^{\cos x}$

(c) $g(x) = e^{-2x^2}$

Example 9: Find the derivative of the following functions.

(a) $f(x) = 5^{3^{x^2}}$

(b) $y = \sin\sqrt{\cos(\cot(3x))}$

Example 10: Find the points on the graph of the function $f(x) = 2\cos x + \cos^2 x$ at which the tangent is horizontal.

Example 11: Find the 100th derivative of $y = \sin(5x)$.

Example 12: The average BAC of eight male subjects was measured after consumption of 15 mL of ethanol. The resulting data were modeled by the concentration function

$$C(t) = 0.0225te^{-0.0467t}$$

where t is measured in minutes after consumption and C is measured in mg/mL.

(a) How rapidly was BAC increasing after 10 minutes?

(b) How rapidly was BAC decreasing half an hour later?

Example 13: A model for the length of day (in hours) in Philadelphia on the t-th day of the year is

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right].$$

Use this model to compare the number of hours of daylight is increasing in Philadelphia on January 15th ($t = 15$) and March 21st ($t = 80$).

Example 14: Use the product rule and chain rule to prove the quotient rule.

Example 15: Find the derivatives of the following functions.

(a) $y = \cos^2(\cot(2x))$

(b) $y = x^3 e^{-1/x^2}$

Example 16: Find an equation of the tangent line to the curve $y = 3^{\sin x}$ at the point where $x = 0$.